Second Semestral Examination
B. Math. (Hons.) IInd year
Rings and Modules
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Answer any SEVEN questions INCLUDING question 9; in each Question, only the first option attempted will be marked. Be BRIEF!

Q 1. If R is a commutative ring such that $R[X]$ is Noetherian, prove that R must have a unity.

OR

Let R be any ring with unity. In the ring $S = M_2(R)$, prove that all two-sided ideals are of the form $M_2(I)$ for some two-sided ideal I of R.

Q 2. Let $\theta : \mathbf{C}[X,Y] \to \mathbf{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^3$. Prove that Ker $\theta = (X^3 - Y^2)$.

OR

Prove that $\mathbb{Z}[\sqrt{-d}]$ is not a UFD if $d > 2$.

Q 3. Determine all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[X]/(X^3 + X^2 + cX + 1)$ is a field.

OR

Let A be a commutative ring with unity, I be an ideal and $P_1, \cdots, P_m$ be prime ideals such that $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_m$. Then show that $I \subseteq P_i$ for some i.

Q 4. Let A be a commutative ring with unity. Prove that the nil radical of $A[X]$ coincides with the Jacobson radical.

OR

Show that for a commutative ring A with unity, if Spec(A) is not connected, then $A \cong A_1 \times A_2$ where the rings A_1, A_2 are both non-zero.

Q 5. For the ring $R = C[0,1]$ of continuous real functions on $[0,1]$, prove: (i) the only idempotents are the constant functions 0 and 1; (ii) R is not Noetherian.

OR

Let $N > 1$ be a positive integer. Show that if $f = \sum_{i=0}^{m} a_i X^i, g = \sum_{j=0}^{n} b_j X^j \in (\mathbb{Z}/N\mathbb{Z})[X]$ are such that $fg = 0$, then $a_i b_j = 0$ for all i, j.

Q 6. If A is a PID and M is a finitely generated A-module, prove that $M/Tor(M)$ is a free module.

OR

Let A be a commutative ring with unity. If I is a finitely generated ideal such that $I = I^2$, show that there is an ideal J such that $A = I \oplus J$.

Q 7. Let A be a commutative ring with unity. and M be a finitely generated A-module. If $\theta : M \to M$ is an onto A-module homomorphism, prove that θ is $1-1$ as well.

OR

If A is a commutative ring and M is a finitely presented A-module, show that for any short exact sequence of A-modules:

$$0 \to K \to A^n \to M \to 0,$$

the module K must be finitely generated.

Q 8. If A is an integral domain, and I, J are ideals such that IJ is a principal ideal, prove that I, J are finitely generated. Is there a counter-example when A is commutative but not an integral domain?

OR

If $A \in M_2(\mathbb{Z})$, consider the group homomorphism $T_A : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $v \mapsto Av$. If $det(A) \neq 0$, find the order and exponent of the group $\mathbb{Z}^2/Image(T_A)$ explicitly in terms of the entries of A.

Q 9. For the matrices $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 5 & 2 \\ -2 & -6 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & -4 & -1 \end{pmatrix}$, find the characteristic polynomials and the Jordan forms.

OR

Find all possible Jordan forms of a matrix whose characteristic polynomial is $(X+2)^2(X-5)^3$.

OR

Define the rational canonical form of a matrix and, prove that any matrix $A \in M_n(K)$ is conjugate to its transpose, where K is a field.